稳定图分层

The points in the boundary of the moduli spaces of pointed, nodal curves with finite automorphism group.

These curves are called stable curves (or pointed stable curves).

1.1 稳定曲线 (Stable Curves)

定义：设 S 是一个概型，$g \geq 2$。一个亏格 g 的稳定曲线 over S 是一个真平坦态射 $\pi : \mathbb{C} \to S$，其几何纤维 $\mathbb{C}_s$ 满足：

- **条件(i)**：$\mathbb{C}_s$ 只有通常双点（ordinary double points）作为奇点
- **条件(ii)**：如果 E 是 $\mathbb{C}_s$ 的一个非奇异有理分量（亏格0），则 E 与其他分量的交点多于2个
- **条件(iii)**：$\dim H^1(\mathcal{O}_{\mathbb{C}_s}) = g$（亏格条件）

关键性质：

- 自同构群有限
- 构成模空间 $\overline{M}_{g,n}$ 的点的几何对象
- 允许节点和标记点，用于紧化模空间

1.2 稳定图 (Stable Graphs)

定义：一个类型为 (G,N) 的稳定图是一个着色无向多重图 $\mathcal{G} = (V,E)$，满足：

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graph LR
    A[稳定图] --> B[顶点 V]
    A --> C[边 E]
    B --> D[颜色: g_v, n_v]
    B --> E[亏格和标记点数]
    C --> F[表示节点]
    C --> G[Legs: 标记点]
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1. 每个顶点 v 有颜色 (g_v, n_v)
2. Γ 连通
3. 总亏格：$\sum_{v\in V} g_v + |E| - (|V| - 1) = G$
4. 总标记点数：$\sum_{v\in V} n_v = N$
5. 稳定性条件：对每个 $g_v = 0$ 的顶点 v，有 $\deg v + n_v \geq 3.$

关于稳定图的定义（取自 STEFANO MAGGIOLO AND NICOLA PAGANI 的 **Generating stable modular graphs**）

Definition 2.2. A *stable graph* of type (G, N) is a colored undirected multigraph $\mathcal{G} = (V, E)$, subject to the following conditions.

(1) The color of a vertex v is given by a pair of natural numbers (g_v, n_v). The two numbers are called respectively the *genus* and the *number of marked points* of the vertex v.
(2) $\mathcal{G}$ is connected.
(3) Its *total genus*, defined as $\sum_{v\in V} g_v + |E| - (|V| - 1)$, equals G.
(4) Its *total number of marked points*, defined as $\sum_{v\in V} n_v$, equals N.
(5) Stability condition: $\deg v + n_v \geq 3$ for every vertex v with $g_v = 0$.

一个稳定图相当于一个稳定曲线的缩影，或者说是一系列相似构造的稳定曲线的缩影，将 irreducible component 分离出来，将他们之间连接的奇异点标记成边，将它们上面的标记点视作连接到外界的 Legs （可以认为是只有一半的边，另一半待连接）

稳定曲线 → 稳定图
不可约分量 → 顶点
节点 → 边

标记点 → 外部边（Legs）

关于一个稳定图如何对应到一个稳定曲线：

（Deligne & Mumford 的 The irreducibility of the space of curves of given genus）

Let Γ be the following (unoriented) graph:

(i) The set of vertices of Γ is the set Γ^0 of irreducible components of X,

(ii) the set of edges of Γ is the set Γ^1 of the singular points of X which lie on two distinct irreducible components,

(iii) an edge $x \in \Gamma^1$ has for extremities the irreducible components on which x lies.

比如说

FIGURE 1. A 2-pointed curve of genus 6 and its dual decorated graph

对于稳定图，我们有如下优美的理论：

引理 1.16：如果 φ 在 Γ 上诱导恒等映射，则 φ 是恒等映射。

推论：稳定图的自同构与曲线的自同构一一对应 ⇒ 稳定图是稳定曲线的"组合缩影"。

Lemma (1.16). — *If* φ *induces the identity on* Γ, *then* φ *is the identity.*

Proof. — If X_1 is an irreducible component of X, then $\varphi(X_1)=X_1$ and φ leaves fixed the points of intersection of X_1 with the other components. In addition, $\mathbf{Pic}^0(X)$ maps onto $\mathbf{Pic}^0(X_1)$ so that φ acts trivially on $\mathbf{Pic}^0(X_1)$. Either:

a) genus $(X_1)\geq 2$ and $\varphi|X_1$ is the identity by Lemma (1.15);

b) genus $(X_1)=1$, φ acts by a translation and leaves a point fixed, so is the identity on X_1;

c) X_1 is the projective line and φ leaves fixed at least three points, so is the identity on X_1. Q.E.D.

- Lemma 1.15

  补充说明：如下引理需要先确保 φ 在 $\mathrm{Pic}^0 X$ 上诱导恒同映射。

  Lemma (1.15). — *If* X *is irreducible, then* φ *is the identity.*

  Proof. — Let φ' be the action of φ on the normalization X' of X. Each singular point of X, together with an ordering of its 2 inverse images in X', defines a distinct morphism from $\mathbf{G}_m$ to $\mathbf{Pic}^0(X)$, so that the inverse image S of the singular locus of X is pointwise fixed by φ'. One has either

  a) $\text{genus}(X')\geq 2$: then conclude by Lemma (1.14);

  b) $\text{genus}(X')=1$, $|S|\geq 2$, then φ' is a translation on X' leaving a point fixed, and so φ' is the identity;

  c) X' is the projective line, $|S|\geq 4$ and φ' is a projectivity leaving more than three points fixed, so is the identity. Q.E.D.

  引理 1.15：如果 X 是不可约的，且 φ 在 $\mathrm{Pic}^0 X$ 上诱导恒同映射，则 φ 是恒等映射。

  证明思路：

  - 考虑正规化 X'
  - 分析不同亏格情况：
    - 亏格 ≥ 2：用引理1.14
    - 亏格 $= 1$：平移固定点 ⇒ 恒等
    - 射影直线：固定至少3点 ⇒ 恒等

这意味着我们对稳定图进行的自同构操作和在曲线上进行自同构操作是一一对应的。

我们几乎可以将稳定图视作一个stable curve 的缩略图。

商映射

定义：设 G 是稳定图，G' 是 G 的商图，且外部边在投影 $\pi : G \to G'$ 中不被收缩。则 G' 有自然稳定图结构：

- 外部顶点：π 下 G 的外部顶点的像
- 顶点亏格：$g(w) = \dim H^1(\pi^{-1}(G)) + \sum_{v\in\pi^{-1}(w)} g(v)$

记为$G' > G$.

命题 3.5：对任何稳定图 G，存在唯一稳定商 $\overline{G} \geq G$，使得：

- $\overline{G}$ 中没有两个亏格0顶点被边连接
- $\overline{G} \leftarrow G$ 只收缩亏格0顶点之间的边

和图的商映射定义相差无几，但是增加了对于亏格的传递，即

$$g(w) = \dim H^1(\pi^{-1}(G)) + \sum_{v\in\pi^1(w)} g(v)$$

（Tosteson）

Proposition 3.5. *Let G be a stable graph. Then there is a unique stable quotient $\overline{G} \geq G$ such that no two distinct genus 0 vertices of $\overline{G}$ are connected by an edge, and $\overline{G} \leftarrow G$ only identifies edges between genus 0 vertices.*

证明思路，

考虑那些亏格为0的点构成的子图，以及这个子图的最小生成树。

考虑商映射将最小生成树变成一个点。

而所有只删了最小生成树的商之间都是同构的

$\mathrm{Stab}(g, n)$ 所有亏格g的有n个标记点的稳定图

$\mathrm{Q}(g, n)$ 所有亏格g的有n个标记点的稳定图中没有任意两个亏格为0的点相连的稳定图

按照自然的想法， 我们有 $\mathrm{Stab}(g,n) \to \mathrm{Q}(g,n); H \mapsto \overline{H}$, 即逐一收缩那些连接两个亏格为0的点的那些边。

但是这个映射并不会保我们最自然的子图

考虑如下的偏序

$$s(J)_0 := \sum_{v\in J, g(v)\geq 1} n(v).$$

The ith entry for $i \geq 1$ is the number of vertices of genus i.

如果删去连接亏格0的边，那么不会变化，如果删去如下三种边，那么就会得到严格大 $s(J) < s(J')$.

(1) a self edge of a single genus g vertex;
(2) an edge between two distinct genus ≥ 1 vertices;
(3) an edge from a genus ≥ 1 vertex to a genus 0 vertex,

Definition 3.4. A *stratification* of a variety X by a poset P is a collection of closed subvarieties $Z(p) \subseteq X$ indexed by $p \in P$, such that if $p \leq q$ then $Z(p) \subset Z(q)$. The subsets $Z(p)$ are called *closed strata*. The *stratum* corresponding to an element $p \in P$ is $S(p) := Z(p) - \bigcup_{q<p} Z(q)$. By construction we have that $Z(p)$ equals the set-theoretic union $\bigcup_{q\leq p} S(q)$.

Stratification

定义 3.4：概型 X 由偏序集 P 的分层是指一簇闭子概型 $Z(p) \subseteq X$ （$p \in P$），满足：

- 如果 $p \leq q$，则 $Z(p) \subset Z(q)$

- 层：$S(p) := Z(p) - \bigcup_{q<p} Z(q)$

应用到模空间：

- $X = \overline{M_{g,n}}$
- $P = (\mathrm{Stab}(g,n), \leq_Q)$ 或 $P = (Q(g,n), \text{quotient})$
- $Z(G) := \bigcup_{J \in \mathrm{Stab}(g,n), J \leq_Q G} \mathcal{M}_J$
- $S(G) = Z(G) - \bigcup_{H \in Q(g,n), H<G} Z(H)$

这样是一个stratification.

Definition 3.9. We use $\overline{\mathcal{M}}_{e,n,0}$ to denote the moduli space of stable curves C of genus e and n marked points, such that all of the irreducible components of C have genus 0. For $G \in \mathrm{Q}(g,n)$ define $\widetilde{S}(G)$ to be the variety

$$\widetilde{S}(G) := \prod_{v \in G,\ g(v)=0} \overline{\mathcal{M}}_{e(v),n(v),0} \times \prod_{v \in G,\ g(v) \geq 1} \mathcal{M}_{g(v),n(v)}.$$

Recall that $e(v)$ denotes the number of self edges of v.

Proposition 3.13. *For $G \in \mathrm{Q}(g,n)$, the map $\widetilde{S}(G) \to S(G)$ induces an isomorphism*

$$\widetilde{S}(G)/A_G \cong S(G).$$

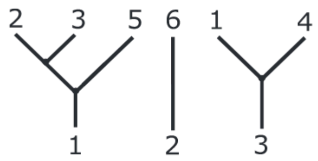
Definition 2.6 (Action of $\mathbf{BT}^{\mathrm{op}}$). Given a stable curve $C \in \overline{\mathcal{M}}_{g,n}$ and a labeled rooted forest $F \in \mathbf{BT}(m,n)$, we define the stable curve $F^*C \in \overline{\mathcal{M}}_{g,m}$ to be $L_F \sqcup_{[m]} C$. In other words F^*C is the curve obtained by gluing L_F to C along the marked points $\{p_i\}_{i \in [m]}$. We use the marking of L_F by $[n]$ to mark F^*C. Since this construction may be performed in families, it corresponds to a map $F^* : \overline{\mathcal{M}}_{g,m} \to \overline{\mathcal{M}}_{g,n}$ for each $F \in \mathbf{BT}(n,m)$, and these maps define an action of $\mathbf{BT}^{\mathrm{op}}$ on $\overline{\mathcal{M}}_{g,n}$.

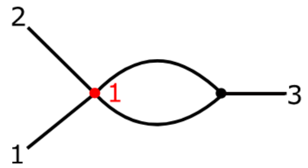
关于为什么这样构造出来的作用是良定义的，其实源自一个过去已经确认的事实，$M_{0,3}$ 是单点集。

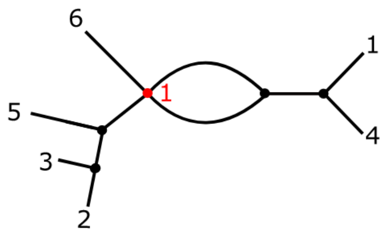
相应的，也就有在稳定图上的一系列作用：

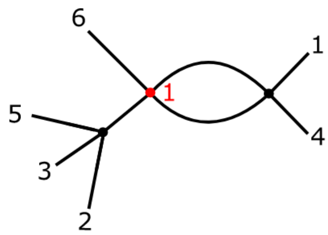
Definition 3.14. The category $\mathbf{BT}^{\mathrm{op}}$ acts on the poset $\mathrm{Stab}(g,n)$ by gluing on trees. More precisely given $F \in \mathbf{BT}(n,m)$ there is a map $\mathrm{Stab}(g,m) \to \mathrm{Stab}(g,n)$, denoted $G \mapsto F^*G$, where F^*G is the stable graph obtained from G by gluing F to G using the identification of the external vertices of G with $[m]$, and erasing bivalent vertices. The action of $\mathbf{BT}^{\mathrm{op}}$ on $\mathrm{Stab}(g,n)$ induces an action of $\mathbf{BT}^{\mathrm{op}}$ on $\mathrm{Q}(g,n)$ in the sense that there is a unique map $F^* : \mathrm{Q}(g,n) \to \mathrm{Q}(g,n)$ such that the diagram

$$\begin{tikzcd}
\mathrm{Stab}(g,n) \arrow[r, "F^*"] \arrow[d, two heads] & \mathrm{Stab}(g,m) \arrow[d, two heads] \\
\mathrm{Q}(g,n) \arrow[r, "F^*"] & \mathrm{Q}(g,m)
\end{tikzcd}$$









Proposition 3.17. *The actions of* $\mathbf{BT}^{\mathrm{op}}$ *on* $\coprod_{G \in \mathrm{Q}(g,n)} S(G)$ *and* $\coprod_{G \in \mathrm{Q}(g,n)} \widetilde{S}(G)$ *induce an* $\mathbf{FS}^{\mathrm{op}}$ *module structure on Borel–Moore homology such that*

$$\bigoplus_{G \in \mathrm{Q}(g,-)} H_\bullet^{\mathrm{BM}}(\widetilde{S}(G)) \to \bigoplus_{G \in \mathrm{Q}(g,-)} H_\bullet^{\mathrm{BM}}(S(G))$$

is a surjection of $\mathbf{FS}^{\mathrm{op}}$ *modules, and the Borel–Moore homology spectral sequence*

$$\bigoplus_{G \in \mathrm{Q}(g,-)} H_\bullet^{\mathrm{BM}}(S(G)) \implies H_\bullet(\overline{\mathcal{M}}_{g,-})$$

is a spectral sequence of $\mathbf{FS}^{\mathrm{op}}$ *modules.*

（BT^{op} 究竟是如何作用在 $S(G)$ 和 $\tilde{S}(G)$ 上的？这个spectral sequences本质上究竟说明了一件什么事？）